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A new approach to the converse of Noether's theorem

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Abstract. The concepts of vector fields and forms along a map are used to establish a condition characterising symmetries of the Hamiltonian system associated with a regular Lagrangian. This condition does not mention any second-order differential equation field but is expressed in terms of the geometry of the second-order tangent bundle. This result is also generalised to the case of Lagrangian functions depending on higher-order derivatives.

1. Introduction and notation

The geometric study of Noether's theorem has contributed greatly to a better understanding of the theorem itself. The Lagrangian approach to this study is a particular case of a more general situation, i.e. symmetries of Hamiltonian systems, in which the infinitesimal symmetries of the Hamiltonian system defined by a regular Lagrangian L are characterised by symmetries of the Lagrangian L. More specifically, if $X \in \mathfrak{X}(M)$, its complete lift $X^c \in \mathfrak{X}(TM)$ satisfies $\mathscr{L}_{X^c}\omega_L = \omega_{X^cL}$ and $\mathscr{L}_{X^c}E_L = E_{X^cL}$, (where $\omega_L = -\mathrm{d}\theta_L$, $\theta_L = \mathrm{d}L \circ S$ and S is the vertical endomorphism on TM (Crampin 1981)) and therefore X^c is a symmetry of the Hamiltonian system (TM, ω_L, E_L) if and only if there exists a closed form $\alpha \in \bigwedge^1(M)$ such that $X^cL = \hat{\alpha}$ (with $\hat{\alpha} \in C^\infty(TM)$ being defined by $\hat{\alpha}(q,v) = \alpha_q(v)$ for $(q,v) \in TM$). In this case, if $\alpha = \mathrm{d}h$, the function $G = \tau^*h - i_{X^c}\theta_L$ is a constant of motion, where τ is the tangent bundle projection.

In a recent paper Marmo and Mukunda (1986) provided a characterisation of all symmetries of (TM, ω_L, E_L), not only complete lifts, in terms of properties of L, by using the set $\mathfrak{X}_D = \{X \in \mathfrak{X}(TM) \mid S([X,D]) = 0\}$ of Newtonoid vector fields with respect to a second-order differential equation (SODE) D and the projection $\pi_D : \mathfrak{X}(TM) \to \mathfrak{X}_D$, given by $\pi_D(X) = X(D) = X + S([D,X])$. In local coordinates, if the vector field X is written $X = \eta^i\partial/\partial q^i + \xi^i\partial/\partial v^i$, then $X(D) = \eta^i\partial/\partial q^i + (D\eta^i)\partial/\partial v^i$. Their results can be summarised as follows.

Theorem. Let $L \in C^\infty(TM)$ be a regular Lagrangian. If $X \in \mathfrak{X}(TM)$ is such that there exists a function $F \in C^\infty(TM)$ satisfying

$$\mathscr{L}_{X(D)}L = \mathscr{L}_D F \qquad \text{for any SODE } D \tag{1}$$

then $G = i_X\theta_L - F$ is a constant of motion. Moreover, if Γ is the dynamical vector field, then $\mathscr{L}_{X(\Gamma)}\omega_L = 0$ and $\mathscr{L}_{X(\Gamma)}E_L = 0$, i.e. $X(\Gamma)$ is a symmetry of the Hamiltonian dynamical system (TM, ω_L, E_L). Conversely, if X is a symmetry of (TM, ω_L, E_L), then $X = X(\Gamma)$ and there exists a function $F \in \mathfrak{X}(TM)$ such that (1) holds.

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Of course, the simplest case is when X is a complete lift $X = Y^c$ with $Y \in \mathfrak{X}(M)$, because then $X(D) = X$ for any SODE D, and F reduces to the pull-back $\tau^* h$ of a function h on the base M.

The point to be remarked is that a vector field X can only be a symmetry of (TM, ω_L, E_L) if it is in $\mathfrak{X}_\Gamma$ and then its vertical components ξ^i are determined by the other ones, namely $\xi^i = \Gamma\eta^i$. If $X_1 - X_2$ is vertical, $X_1(D) = X_2(D)$, and therefore it is not a specific X but an equivalence class 'up to a vertical field' which is playing the relevant role in establishing the constant of the motion. This is better displayed by means of the concept of section along a map, which has also been used recently (Gracia and Pons 1989) for an alternative geometric interpretation of the time-evolution operator for singular Lagrangians (Cariñena and López 1987), that we introduce in the next section.

We will make use of the theory of higher-order tangent bundles, and we generally follow the notation of the paper by Crampin *et al* (1986): $\tau_{k,l}: T^k M \to T^l M$ are the bundle projections, $T^{(k)}$ is the map given the total time derivative $d_{T^{(k)}}$, $f_{(n)}$ are the functions given by

$$f_{(n)}(Q) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\{f \circ \sigma\}\bigg|_{t=0}$$

for $Q \in T^k M$ and σ a representative of Q, etc. We will also make use of the theory of ρ-derivations (see Pidello and Tulczyjew 1987).

2. Sections along a map

Let $\pi: E \to M$ be a fibre bundle, $f: N \to M$ a C^∞-differentiable map and denote $f^* E$ the pull-back of the bundle E by the map f. By a section of E along the map f (or over f) we mean a C^∞-map $\sigma: N \to E$ such that $\pi \circ \sigma = f$. There exists a one-to-one canonical correspondence between the set of sections along f and sections of the bundle $f^* E$ over N. Moreover, if E is a vector bundle, then both sets can be endowed with $C^\infty(N)$-module structures and this correspondence is an isomorphism of $C^\infty(N)$-modules. For details, see e.g. Poor (1981). The most important cases for our proposals are when the vector bundle is either TM or $(T^*M)^{\wedge p}$, $(T^*M)^{\otimes p} \otimes (TM)^{\otimes q}$, or something similar, and the sections along f are then said to be vector fields, p-forms or (p, q)-type tensor fields along f, and will be denoted $\mathfrak{X}(f)$, $\bigwedge^p(f)$ or $\mathfrak{T}^{p,q}(f)$, respectively.

Two special instances of vector fields over f are the restriction of a vector field $X \in \mathfrak{X}(M)$ to $f(N)$ obtained by composition of $f: N \to M$ and $X: M \to TM$ (usually denoted $X \circ f$), and the image of a vector field $Y \in \mathfrak{X}(N)$ under f, denoted $f_* \circ Y$ and given by $(f_* \circ Y)(n) = f_{*n}(Y(n))$. The more traditional concept of vector field in M arises here as a vector field along the identity map in the base M, i.e. $\mathfrak{X}(M) = \mathfrak{X}(\mathrm{id}_M)$, and in a similar way, $\bigwedge^p(M) = \bigwedge^p(\mathrm{id}_M)$ and $\mathfrak{T}^{p,q}(M) = \mathfrak{T}^{p,q}(\mathrm{id}_M)$.

The set of vector fields over f is endowed with a $C^\infty(N)$-module structure: they can be added to each other and multiplied by $C^\infty(N)$ functions and in particular, if $(\mathscr{V}, z)$ and $(\mathscr{U}, x)$ are charts in N and M, respectively, such that $f(\mathscr{V}) \subset \mathscr{U}$, X is a vector field along f and $n \in \mathscr{V}$, the expression of X_n in these coordinates is

$$X_n = \xi^i(n) \frac{\partial}{\partial x^i}\bigg|_{f(n)}$$

so that every vector field X along f can be written as a linear combination

$$X = \xi^i \left(\frac{\partial}{\partial x^i} \circ f \right) \tag{2}$$

with functions in $C^\infty(N)$ as coefficients. In the same way a p-form α along f has a local expression

$$\alpha = \alpha_{i_1 \dots i_p} (\mathrm{d}x^{i_1} \circ f) \wedge \cdots \wedge (\mathrm{d}x^{i_p} \circ f) \tag{3}$$

where $\alpha_{i_1 \dots i_p} \in C^\infty(N)$. Moreover, $\mathfrak{X}(f)$ and $\bigwedge^1(f)$ are dual moduli by means of the pairing $\langle X, \alpha \rangle (n) = \langle X_n, \alpha_n \rangle$. We sometimes put $\alpha(X) = \langle X, \alpha \rangle$.

It has been shown by Pidello and Tulczyjew (1987) that a vector field X along f determines two $f^\star$-derivations of scalar forms on M: one of type $i_\star$ and degree -1, denoted i_X, and other of type $d_\star$, denoted d_X. If $g: P \to N$ is another differentiable map, then

$$g^\star \circ i_X = i_{X \circ g} \qquad \text{and} \qquad g^\star \circ d_X = d_{X \circ g}. \tag{4}$$

In the following we will be concerned with cases in which f are the natural projections $\tau_{l,k}: T^l M \to T^k M$, for k and l non-negative integer numbers such that $l > k$ and particularly the case $l = k + 1$. We recall (Crampin *et al* 1986) that there is a natural map

$$\boldsymbol{T}^{(k)} : T^{k+1} M \to T(T^k M) \tag{5}$$

where if $Q \in T^{k+1} M$ then $\boldsymbol{T}^{(k)}(Q)$ is the vector at $\tau_{k+1,k}(Q)$ tangent to the curve $\tilde{\rho}^k : I \subset R \to T^k M$ lifted from a curve $\rho : I \subset R \to M$ in the equivalence class defining Q. This map can also be considered as a vector field along $\tau_{k+1,k}$ according to the preceding definition.

3. Liftings of vector fields along projections

Let X be a vector field along $\tau_{1,0}$. For every $k > 0$ there exists one vector field denoted $X^{(k)}$ over $\tau_{k+1,k}$ such that

$$X^{(k-1)} \circ \tau_{k+1,k} = \tau_{k,k-1\star} \circ X^{(k)} \tag{6}$$

and satisfies

$$d_{X^{(k)}} \circ d_{\mathbf{T}^{(k-1)}} = d_{\mathbf{T}^{(k)}} \circ d_{X^{(k-1)}} \tag{7}$$

where, for $k = 1$, $X^{(0)} = X$.

We will give an explicit construction of such vector field. Let us choose $\bar{X} \in \mathfrak{X}(TM)$ a vector field on TM such that $\tau_{1,0\star} \circ \bar{X} = X$. For every $Q \in T^{k+1} M$, let ρ be a curve in M representative of Q and $\tilde{\rho}^k$ the lifted curve in $T^k M$. Then the map $\chi : \mathscr{U} \subset R^2 \to M$ defined by

$$\chi(s,t) = (\tau_{1,0} \circ \bar{\phi}_s \circ \tilde{\rho}^k)(t)$$

where $\bar{\phi}_s$ is the flow of $\bar{X}$, can be used to determine a vector $X_Q^{(k)}$ tangent to T^kM. More specifically, for any real number s, the curve in M, $\chi_s(t) = \chi(s,t)$, defines a point in T^kM. As a function of the parameter s we obtain a curve in T^kM whose tangent vector for $s=0$ is the above-mentioned vector $X_Q^{(k)}$.

This tangent vector does not depend either on the vector field $\bar{X}$, or on the choice of the curve representative ρ of Q. A straightforward calculation shows that for any function $f \in C^\infty(M)$ and $n = 0,\ldots,k$

$$\frac{\partial^n}{\partial t^n}\{f \circ \chi\}\bigg|_{(0,0)} = \frac{\partial^n}{\partial t^n}\{f \circ \rho\}\bigg|_{t=0} \tag{8}$$

and

$$\frac{\partial^{n+1}}{\partial s \partial t^n}\{f \circ \chi\}\bigg|_{(0,0)} = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\{(Xf) \circ \tilde{\rho}^1\}|_{t=0}. \tag{9}$$

We can see from (8) that $X_Q^{(k)}$ is tangent to T^kM at the point $\tau_{k+1,k}(Q)$, so that the map $X^{(k)}: Q \mapsto X_Q^{(k)}$ is a vector field along $\tau_{k+1,k}$. Now (9) shows that

$$X_Q^{(k)} f_{(n)} = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\{(Xf) \circ \tilde{\rho}^1\}\bigg|_{t=0} \tag{10}$$

and from the definition of $\boldsymbol{T}^{(s)}$ we have

$$X^{(k)} f_{(n)} = (d_{\mathrm{T}^{(n)}} \circ \cdots \circ d_{\mathrm{T}^{(1)}})(d_X f) \tag{11}$$

which implies (6) and (7).

Since two vector fields agreeing over functions of the type $f_{(n)}$ must be equal, we can conclude that $X^{(k)}$ is uniquely defined.

In the particular case of X being the restriction of a vector field $Y \in \mathfrak{X}(M)$, $X = Y \circ \tau_{1,0}$, the vector field $X^{(k)}$ reduces to the restriction of the complete lift $Y^{c,k} \in \mathfrak{X}(T^kM)$, namely $X^{(k)} = Y^{c,k} \circ \tau_{k+1,k}$. This is the reason why we call $X^{(k)} \in \mathfrak{X}(\tau_{k+1,k})$ the *generalised complete lift* of $X \in \mathfrak{X}(\tau_{1,0})$.

If the coordinate expression for X is

$$X = \eta^i \left(\frac{\partial}{\partial q^i} i \circ \tau_{1,0}\right) \tag{12}$$

then the corresponding expression for $X^{(k)}$ is

$$X^{(k)} = \sum_{n=0}^{k} \eta^i_{[n]} \left(\frac{\partial}{\partial q^i_{(n)}} \circ \tau_{k+1,k}\right) \tag{13}$$

where $\eta^i_{[n]} = \tau_{k+1,n+1}{}^\star(d_{\mathrm{T}^{(n)}} \circ \cdots \circ d_{\mathrm{T}^{(1)}})\eta^i$, for $n = 1,\ldots,k$, and $\eta^i_{[0]} = \tau_{k+1,1}{}^\star\eta^i$.

Property (7) is equivalent to

$$i_{X^{(k-1)}} \circ d_{\mathrm{T}^{(k)}} - d_{\mathrm{T}^{(k-1)}} \circ i_{X^{(k)}} = 0 \tag{14}$$

because both terms vanish on $C^\infty(T^kM)$, while on exact forms: $\mathrm{d}F \in \bigwedge^1(T^kM)$,

$$\begin{aligned} i_{X^{(k-1)}} \circ d_{\mathrm{T}^{(k)}}\mathrm{d}F - d_{\mathrm{T}^{(k+1)}} \circ i_{X^{(k)}}\mathrm{d}F &= \left(i_{X^{(k-1)}} \circ \mathrm{d} \circ d_{\mathrm{T}^{(k)}} - d_{\mathrm{T}^{(k+1)}} \circ i_{X^{(k)}} \circ \mathrm{d}\right)F \\ &= \left(d_{X^{(k-1)}} \circ d_{\mathrm{T}^{(k)}} - d_{\mathrm{T}^{(k+1)}} \circ d_{X^{(k)}}\right)f = 0. \end{aligned} \tag{15}$$

Semibasic forms can be identified with forms along the projection map. We consider only the projection $\tau_{1,0}$, but the same is true for the other ones. For instance if $\alpha \in \bigwedge^1(TM)$ is semibasic then the associated $\check{\alpha} \in \bigwedge^1(\tau_{1,0})$ is defined by

$$\check{\alpha}_Q(v) = \alpha_Q(v^\dagger)$$

where $Q \in TM$, $v \in T_{\tau_{1,0}(Q)}M$ and $v^\dagger \in T_Q(TM)$ is such that $\tau_{1,0*Q}v^\dagger = v$. This fact implies that if $X \in \mathfrak{X}(\tau_{1,0})$ and $\tilde{X} \in \mathfrak{X}(\tau_{2,1})$ are such that $\tau_{1,0*} \circ \tilde{X} = X \circ \tau_{2,1}$, then

$$i_{\tilde{X}}\alpha = \tau_{2,1}{}^*\check{\alpha}(X). \tag{16}$$

As a consequence we have that for all $X \in \mathfrak{X}(\tau_{1,0})$ and $\alpha \in \bigwedge^1(TM)$

$$d_{\mathbf{T}^{(1)}}(\check{\alpha}(X)) = (d_{\mathbf{T}^{(1)}}\alpha)^\vee(X^{(1)}) \tag{17}$$

which will be used later on. Indeed,

$$\tau_{3,2}{}^* d_{\mathbf{T}^{(1)}}\check{\alpha}(X) = d_{\mathbf{T}^{(2)}}\tau_{2,1}{}^*\check{\alpha}(X) = d_{\mathbf{T}^{(2)}}i_{X^{(1)}}\alpha = i_{X^{(2)}}d_{\mathbf{T}^{(1)}}\alpha$$

and as $d_{\mathbf{T}^{(1)}}\alpha$ is semibasic over TM we can put $i_{X^{(2)}}d_{\mathbf{T}^{(1)}}\alpha = \tau_{3,2}{}^*(d_{\mathbf{T}^{(1)}}\alpha)^\vee(X^{(1)})$. Since the map $\tau_{3,2}$ is a submersion it follows that (17) holds.

4. Second-order differential equations

There are two alternative but equivalent geometrical interpretations for a time-independent second-order differential equation in M. It can be seen either as a special vector field $\Gamma \in \mathfrak{X}(TM)$ projecting on $\boldsymbol{T}^{(0)}$:

$$\tau_{TM*} \circ \Gamma = \boldsymbol{T}^{(0)} \tag{18}$$

or as a section $\gamma : TM \to T^2M$ for $\tau_{2,1}$. The relation between Γ and γ is given by

$$\Gamma = \boldsymbol{T}^{(1)} \circ \gamma \tag{19}$$

or in an equivalent way by

$$\mathscr{L}_\Gamma = \gamma^* \circ d_{\mathbf{T}^{(1)}} \tag{20}$$

as operators on $\bigwedge(TM)$.

From (18) we see that for any function $f \in C^\infty(M)$ and any SODE Γ,

$$d_{\mathbf{T}^{(0)}}f = \mathscr{L}_\Gamma(\tau_{1,0}{}^*f). \tag{21}$$

On the other hand, the set $\mathfrak{X}_\Gamma$ of vector fields (Marmo and Mukunda 1986, Sarlet 1987, Cariñena *et al* 1989b), as well as other related sets, has been shown to play a relevant role in both the study of symmetries of Γ and the analysis of the so-called inverse problem of Lagrangian mechanics. Essentially, the vector fields in the set $\mathfrak{X}_\Gamma$

are those vector fields in TM preserving the second-order character of Γ. We are now going to analyse the relation of this set to the generalised complete lifts defined above.

There exists a one-to-one correspondence betwen $\mathfrak{X}(\tau_{1,0})$ and $\mathfrak{X}_\Gamma$, given by

$$\begin{aligned} I_\Gamma &: \mathfrak{X}(\tau_{1,0}) \longrightarrow \mathfrak{X}_\Gamma \\ X &\longmapsto X^{(1)} \circ \gamma \end{aligned} \tag{22}$$

whose inverse is the restriction $\tau_{1,0*}\big|_{\mathfrak{X}_\Gamma}$ of $\tau_{1,0*}$ onto $\mathfrak{X}_\Gamma$. Let X be a vector field along $\tau_{1,0}$ and γ the section of $\tau_{2,1}$ associated to Γ. Then, since for any $Q \in TM$ the tangent vector $X^{(1)}_{\gamma(Q)}$ is in $T_Q TM$, we have that $X^{(1)} \circ \gamma$ is a vector field in TM. Moreover, this vector field projects on X:

$$\tau_{1,0*} \circ X^{(1)} \circ \gamma = X \circ \tau_{2,1} \circ \gamma = X \tag{23}$$

which shows that $\tau_{1,0*} \circ I_\Gamma = \mathrm{id}_{\mathfrak{X}(\tau_{1,0})}$. Then, bearing in mind (19), (21) and (7) we have that, for $f \in C^\infty(M)$,

$$\begin{aligned} \left[(d_{X^{(1)}} \circ d_{\mathbf{T}^{(0)}} - d_{\mathbf{T}^{(1)}} \circ d_{X^{(0)}})f\right](\gamma(Q)) &= X^{(1)}_{\gamma(Q)}(\boldsymbol{T}^{(0)}f) - \boldsymbol{T}^{(1)}_{\gamma(Q)}(Xf) \\ &= (X^{(1)} \circ \gamma)_Q(\Gamma(\tau_{1,0}{}^*f)) - \Gamma_Q((X^{(1)} \circ \gamma)(\tau_{1,0}{}^*f)) \\ &= [X^{(1)} \circ \gamma, \Gamma](\tau_{1,0}{}^*f) \end{aligned} \tag{24}$$

vanishes or, written in a different but equivalent way, $S([\Gamma, X(\Gamma)]) = 0$, i.e. the condition for $X^{(1)} \circ \gamma$ to be in $\mathfrak{X}_\Gamma$.

In order to see that $\tau_{1,0*}\big|_{\mathfrak{X}_\Gamma}$ is the inverse of I_Γ, let Y be a vector field in $\mathfrak{X}_\Gamma$ and denote $X = \tau_{1,0*} \circ Y \in \mathfrak{X}(\tau_{1,0})$. Then by (23) we have

$$X^{(1)}_{\gamma(Q)} f_{(0)} = X_Q f = Y_Q f_{(0)} \tag{25}$$

so that $X^{(1)} \circ \gamma - Y$ is a vertical vector field. Since both, $X^{(1)} \circ \gamma$ and Y, are in $\mathfrak{X}_\Gamma$ they must be equal.

5. Noether's theorem

Let L be a regular Lagrangian in TM. The Euler–Lagrange equations for this Lagrangian can be written $i_\Gamma \omega_L = \mathrm{d}E_L$, or equivalently $\mathscr{L}_\Gamma \theta_L - \mathrm{d}L = 0$, since L is regular. Alternatively, if $\gamma: TM \to T^2M$ is the section corresponding to Γ , then the Euler–Lagrange equations can be written as $\gamma^*(d_{\mathbf{T}^{(1)}}\theta_L - \tau_{2,1}{}^*\mathrm{d}L) = 0$. The 1-form $\delta L = d_{\mathbf{T}^{(1)}}\theta_L - \tau_{2,1}{}^*\mathrm{d}L$ in T^2M is called the Euler–Lagrange 1-form. It is a semibasic form over M, so that $(\delta L)^\vee$ is a 1-form along $\tau_{2,0}$. The Cartan form $\theta_L \in \bigwedge^1(TM)$ is also semibasic over M so that $\check{\theta}_L$ is a 1-form over $\tau_{1,0}$.

Let $X \in \mathfrak{X}(\tau_{1,0})$. We have the following identity:

$$d_{X^{(1)}} L = -(\delta L)^\vee(X) + d_{\mathbf{T}^{(1)}}(\check{\theta}_L(X)) \tag{26}$$

which can be proved as follows. Using (7) for $k = 1$ we obtain

$$\tau_{3,2}{}^* d_{X^{(1)}} L = d_{X^{(2)}} \tau_{2,1}{}^* L = i_{X^{(2)}} \tau_{2,1}{}^* \mathrm{d}L.$$

Then

$$\tau_{3,2}{}^{*}d_{X^{(1)}}L = -i_{X^{(2)}}(d_{\mathbf{T}^{(1)}}\theta_L - \tau_{2,1}{}^{*}\mathrm{d}L) + i_{X^{(2)}}d_{\mathbf{T}^{(1)}}\theta_L$$
$$= \tau_{3,2}{}^{*}\left(-(\delta L)^{\vee}(X) + d_{\mathbf{T}^{(1)}}(\check{\theta}_L(X))\right)$$

where we have used (17). Then, as $\tau_{3,2}$ is a submersion, (26) follows.

If $F \in C^{\infty}(TM)$ we have

$$d_{X^{(1)}}L - d_{\mathbf{T}^{(1)}}F = -(\delta L)^{\vee}(X) + d_{\mathbf{T}^{(1)}}(\check{\theta}_L(X) - F) \tag{27}$$

so that if there exist $F \in C^{\infty}(TM)$ such that

$$d_{X^{(1)}}L - d_{\mathbf{T}^{(1)}}F = 0$$

then the function $G = \check{\theta}_L(X) - F$ satisfies $d_{\mathbf{T}^{(1)}}G = (\delta L)^{\vee}(X)$; that is, G is a first integral of Γ.

Conversely, if G is a constant of the motion given by Γ, then there will be $X \in \mathfrak{X}(\tau_{1,0})$ such that

$$d_{\mathbf{T}^{(1)}}G = (\delta L)^{\vee}(X)$$

and then, defining F by $F = \check{\theta}_L(X) - G$, it follows from equation (27) that

$$d_{X^{(1)}}L - d_{\mathbf{T}^{(1)}}F = 0.$$

These results can be summarised in the following version of Noether's theorem.

Theorem. Let X be a vector field along $\tau_{1,0}$ and L a regular Lagrangian. If there exists a function $F \in C^{\infty}(TM)$ such that

$$d_{X^{(1)}}L = d_{\mathbf{T}^{(1)}}F \tag{28}$$

then the function $G = F - \check{\theta}_L(X)$ is a constant of motion. Conversely if G is a first integral of the motion given by L, then there exist $X \in \mathfrak{X}(\tau_{1,0})$ and $F \in C^{\infty}(TM)$ such that (28) holds.

Condition (28) is equivalent to that of Marmo and Mukunda (1986) but it does not make reference to any specific SODE which is an important advantage as far as the possible generalisation to higher-order mechanics and classical field theories is concerned. Actually, if D is an arbitrary SODE and δ denotes the corresponding section $\delta : TM \to T^2M$, related by $D = \delta^{*} \circ d_{\mathbf{T}^{(1)}}$, the δ-pull-back of the equation (28) gives

$$\mathscr{L}_{X(D)}L - \mathscr{L}_{D}F = 0$$

and we obtain in this way the condition given in Marmo and Mukunda (1986). Conversely, if this condition holds for every SODE D, given an arbitrary point $Q \in T^2M$, it is possible to choose a SODE D such that $Q = \delta(\tau_{2,1}(Q))$ and therefore $(d_{X^{(1)}}L - d_{\mathbf{T}^{(1)}}F)(Q) = 0$. This property being true for any $Q \in T^2M$, condition (28) is recovered.

6. Generalisation to higher-order mechanics

In this section we wish to show how the results obtained in the last section can be extended to higher-order mechanics. First, we need a generalisation of the construction of §3 to the case of vector fields along $\tau_{l,0}$ with l a natural number. The details of this construction will appear elsewhere (Cariñena *et al* 1989c); we only say here that the the liftings $X^{(r)}$ of a vector field $X \in \mathfrak{X}(\tau_{l,0})$ are characterised by the following two conditions:

$$X^{(k-1)} \circ \tau_{k+l,k+l-1} = \tau_{k,k-1\star} \circ X^{(k)} \tag{29}$$

and

$$d_{X^{(k)}} \circ d_{\mathrm{T}^{(k-1)}} = d_{\mathrm{T}^{(k+l-1)}} \circ d_{X^{(k-1)}}. \tag{30}$$

In higher-order mechanics, the Lagrangian is a function on T^kM so that (assuming that L is regular) the Euler–Lagrange equations define in a unique way a $2k$ order ordinary differential equation, that is, a vector field $\Gamma \in \mathfrak{X}(T^{2k-1}M)$ projecting on $T^{(2k-2)}$ or alternatively a section γ of the bundle $\tau_{2k,2k-1}: T^{2k}M \to T^{2k-1}M$. More precisely, the Euler–Lagrange equations are $\mathscr{L}_\Gamma\theta_L - \tau_{2k-1,k}{}^\star\mathrm{d}L = 0$ or alternatively $\gamma^\star\delta L = 0$ where $\delta L = d_{\mathrm{T}^{(2k-1)}}\theta_L - \tau_{2k,k}{}^\star\mathrm{d}L$. (See Crampin *et al* (1986) for the details of the construction of θ_L.)

Proposition. Let X be a vector field along $\tau_{2k-1,0}$, $X \in \mathfrak{X}(\tau_{2k-1,0})$. Then the following property holds:

$$d_{X^{(k)}}L = d_{\mathrm{T}^{(3k-2)}}(\check{\theta}_L(X^{(k-1)})) - \tau_{3k-1,2k}{}^\star[(\delta L)^\vee(X)]. \tag{31}$$

Proof. Since $\delta L = d_{\mathrm{T}^{(2k-1)}}\theta_L - \tau_{2k,k}{}^\star\mathrm{d}L$, and using properties (29) and (30), we have

$$\begin{aligned} i_{X^{(2k)}}\delta L &= i_{X^{(2k)}}[d_{\mathrm{T}^{(2k-1)}}\theta_L - \tau_{2k,k}{}^\star\mathrm{d}L] \\ &= i_{X^{(2k)}}d_{\mathrm{T}^{(2k-1)}}\theta_L - \tau_{4k-1,3k-1}{}^\star d_{X^{(k)}}L \\ &= d_{\mathrm{T}^{(4k-2)}}i_{X^{(2k-1)}}\theta_L - \tau_{4k-1,3k-1}{}^\star d_{X^{(k)}}L. \end{aligned}$$

Now, taking into account that δL is semibasic over M and θ_L is semibasic over $T^{k-1}M$, we can put

$$i_{X^{(2k)}}\delta L = \tau_{4k-1,2k}{}^\star(\delta L)^\vee(X)$$

and

$$i_{X^{(2k-1)}}\theta_L = \tau_{4k-2,3k-2}{}^\star\check{\theta}_L(X^{(k-1)})$$

so that

$$\tau_{4k-1,3k-1}{}^\star\{d_{X^{(k)}}L - d_{\mathrm{T}^{(3k-2)}}(\check{\theta}_L(X^{(k-1)})) + \tau_{3k-1,2k}{}^\star[(\delta L)^\vee(X)]\} = 0$$

and since $\tau_{4k-1,3k-1}$ is a submersion we get the above-mentioned result.

Let us suppose that there exist a vector field $X \in \mathfrak{X}(\tau_{2k-1,0})$ and a function $F \in C^\infty(T^{3k-2}M)$ such that

$$d_{X^{(k)}} L = d_{\mathrm{T}^{(3k-2)}} F .$$

Then, from (31) we have

$$d_{\mathrm{T}^{(3k-2)}} [F - \check{\theta}_L(X^{(k-1)})] = -\tau_{3k-1,2k}{}^*[(\delta L)^\vee(X)]$$

so that there exists a funtion $G \in C^\infty(T^{2k-1}M)$ such that $\tau_{3k-2,2k-1}{}^* G = F - \check{\theta}_L(X^{(k-1)})$, i.e. $d_{\mathrm{T}^{(2k-1)}} G = -(\delta L)^\vee(X)$, and composing with γ^* we have that G is a constant of the motion.

Conversely let $G \in C^\infty(T^{2k-1}M)$ be a constant of the motion. Since $d_{\mathrm{T}^{(2k-1)}} G$ vanishes along the equations of motion, there will exist a vector field X along $\tau_{2k-1,0}$ such that $d_{\mathrm{T}^{(2k-1)}} G = -(\delta L)^\vee(X)$, and then defining $F = \check{\theta}_L(X^{(k-1)}) - \tau_{3k-2,2k-1}{}^* G$, the equation (31) becomes $d_{X^{(k)}} L = d_{\mathrm{T}^{(3k-2)}} F$.

Then, the generalisation of the above version of Noether's theorem for these higher-order Lagrangians is as follows.

Theorem. Let L be a Lagrangian function in T^kM and X a vector field along $\tau_{2k-1,0}$. If there exists a function $F \in C^\infty(T^{3k-2}M)$ such that $d_{X^{(k)}} L = d_{\mathrm{T}^{(3k-2)}} F$ then the function $G \in C^\infty(T^{2k-1}M)$ defined by $\tau_{3k-2,2k-1}{}^* G = F - \check{\theta}_L(X^{(k-1)})$ is a constant of the motion. Conversely, given a constant of the motion $G \in C^\infty(T^{2k-1}M)$ there exists a vector field X along $\tau_{2k-1,0}$ and a function F in $T^{3k-2}M$ such that $d_{X^{(k)}} L = d_{\mathrm{T}^{(3k-2)}} F$.

In summary, we have been able to give a generalisation of Noether's theorem admitting a converse, by considering as fundamental objects vector fields and forms along a map, which are seldom found in the physics literature, instead of true vector fields and forms. The usefulness of these concepts will be shown in a subsequent paper (Cariñena *et al* 1989c) where their role in the geometric foundation of clasical mechanics will be remarked upon.

Noether's theorem is here presented not only as an academic reformulation of Marmo and Mukunda (1986) but in such a way that it admits a straightforward generalisation for higher-order mechanics, as was shown in §6. Furthermore, in the case of first-order Lagrangians, these concepts are used to establish a one-to-one correspondence between $\mathfrak{X}$ $(\tau_{1,0})$ and the set $\mathfrak{X}_\Gamma$ of Newtonoid vector fields for every second-order differential equation field Γ by taking advantage of a twofold geometric interpretation of the second-order differential equations.

Finally, it is worthy of mention that this theorem is directly generalisable to classical field theory. There, the role of the canonical vector field $\boldsymbol{T}^{(k)}$ is played by the horizontalisation operator $h^{(k)}$, the total time derivative becomes a total divergence of a vector field along π_1 given by $d_{h^{(1)}}(i_X\Omega) = \mathrm{div}(X)\Omega$, where Ω is a volume form in the base, and the role of the symplectic structure is played by the multisymplectic structure defined by the Lagrangian function (see Saunders (1989) and Cariñena *et al* (1989a) for the notation).

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